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Convergence of Cubic Spline Interpolants of Functions Possessing Discontinuities

G. D. ANDRIA, G. D. BYRNE, AND C. A. HALL

Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15213

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1. INTRODUCTION

Let $\{\Pi_n\}$ be a sequence of partitions of $[-1, 1]$,

$$\Pi_n : -1 = x_0^{(n)} < x_1^{(n)} < \cdots < x_n^{(n)} = 1.$$

Let $\{S_n f\}$ be the sequence of *cubic spline interpolants* associated with $\{\Pi_n\}$ and a function f defined on $[-1, 1]$. Then $S_n f$ is cubic in each interval $[x_i^{(n)}, x_{i+1}^{(n)}]$, $S_n f$ belongs to $C^2[-1, 1]$, $S_n f(x_i^{(n)}) = f(x_i^{(n)})$, $i = 0,\ldots n$, and $(S_n f)'(-1) = (S_n f)'(1) = 0$. In [7] one of us established that if $f \in C[-1, 1]$,

$$P_n = \max_{|i-j|=1} \frac{h_i^{(n)}}{h_j^{(n)}} < P \qquad \text{for all } n, \tag{1}$$

and

$$Q_n = \max_i \left| \frac{h_i^{(n)}}{h_{i+1}^{(n)} + h_{i+2}^{(n)}} ; \frac{h_{i+1}^{(n)}}{h_{i-1}^{(n)} + h_i^{(n)}} \right| < Q < 2 \tag{2}$$

for all n, where $h_j^{(n)} = x_j^{(n)} - x_{j-1}^{(n)}$, then

$$\max_{|x|\leq 1} |f - S_n f| \leq \frac{4 + 3P + 2Q}{2(2 - Q)} \omega(f; \delta_n), \tag{3}$$

where $\omega(f; \delta_n)$ is the modulus of continuity of f and $\delta_n = \max_i h_i^{(n)}$ is the mesh gauge. Consequently, $S_n f \to f$ uniformly on $[-1, 1]$, if $\delta_n \to 0$. For other such convergence results see [2, 3, 5, 8, 9].

In this paper we investigate the behavior of $\{S_n f\}_{n=1}^{\infty}$ if f has a jump discontinuity at $x = 0$. We establish (Theorem 1) pointwise convergence of $S_n f(x)$ to $f(x)$ if $\delta_n \to 0$, for all points $x \neq 0$. Golomb [5, Theorem 5] has established L_2-convergence of $S_n f$ to f under stronger conditions on f and for uniform meshes $\{\Pi_n\}$ only. See also Swartz and Varga [10].

We also show (Theorem 2) uniform convergence on $[-1, 1]$ of $S_n f$ if f' has a finite number of jump discontinuities. Both Theorems 1 and 2 present explicit error bounds as well as asymptotic behavior.

2. Convergence Theorems

First some notation. Let $I = [-1, 1]$, $J = [-1, -a/4] \cup [a/4, 1]$, $K = \overline{I - J}$, $L = [-1, -a/2] \cup [a/2, 1]$, where a is specified below. Let $\|u\|_M = \sup\{|u(x)| : x \in M\}$ and $\omega_M(u; h) = \sup\{|u(x_2) - u(x_1)| : x_i \in M$ and $|x_2 - x_1| \leq h\}$ for $M = I, J, K$ or L. Also set $K_0 = K - \{0\}$ and $I_0 = I - \{0\}$.

Our first result establishes that introducing a discontinuity at $x = 0$ in the function does *not* destroy the pointwise convergence of $\{S_n f(x)\}_{n=1}^{\infty}$ to $f(x)$ at points where f is continuous.

Theorem 1. *Let $f \in C(I_0)$ and be bounded on I. Choose $\{\Pi_n\}$ so as to satisfy* (1) *and* (2). *For an arbitrary* (*but fixed*) $a \in (0, 1)$ *the cubic spline interpolant $S_n f$ satisfies*

$$|(f - S_n f)(x)| \leq K_1 \delta_n + K_2 \left\{\omega_J(f, \delta_n) + \frac{2\delta_n \Delta_a}{a}\right\} \qquad \textit{for} \quad |x| \geq a/2, \tag{4}$$

where

$$K_1 = 6(1 + P) \max\left\{\frac{2}{2 - Q}, 3\right\} \|f\|_K,$$

$$K_2 = \frac{4 + 3P - 2Q}{2(2 - Q)}, \qquad \textit{and} \qquad \Delta_a = |f(a/4) - f(-a/4)|.$$

Hence for all $x \neq 0$, $\{S_n f(x)\}$ converges to $f(x)$ if $\delta_n \to 0$.

Proof. Fix n and delete the superscripts for ease of notation. Let

$$i_1 = \min\{i \mid x_i \geq -a/2\}, \; i_2 = \max\{i \mid x_i \leq a/2\}.$$

Further, define $g \in C[-1, 1]$ by

$$\begin{aligned} g(x) &= f(x), \qquad x \in J, \\ &= f(-a/4) + \frac{2}{a}(f(a/4) - f(-a/4))(x + a/4), \qquad x \in K. \end{aligned}$$

If $S_n g$ is the cubic spline interpolant of g, then from [7, Theorem 2].[1]

$$\|g - S_n g\|_I \leq \frac{4 + 3P - 2Q}{2(2 - Q)} \omega_I(g; \delta_n). \tag{5}$$

[1] This bound was improved in the final form of [7, Theorem 2].

But $\omega_J(g;\delta_n)=\omega_J(f;\delta_n)$ and $\omega_K(g;\delta_n)\leqslant \max_{x\in K}|g'(x)|\,\delta_n=2(\delta_n/a)|f(a/4)-f(-a/4)|$. Thus

$$\|g-S_ng\|_I\leqslant \frac{4+3P-2Q}{2(2-Q)}\left[\omega_J(f;\delta_n)+\frac{2\delta_n\Delta_a}{a}\right]. \tag{6}$$

Now consider the spline $s=S_ng-S_nf$ which satisfies $s(x_j)=0$ for $x_j\in J$, $s(x_j)=g(x_j)-f(x_j)$ for $x_j\in K$ and $s'(-1)=s'(1)=0$. For $x\in L$

$$\begin{aligned}|(f-S_nf)(x)|&=|(g-S_nf)(x)|\\&\leqslant|(g-S_ng)(x)|+|(S_ng-S_nf)(x)|.\end{aligned}\tag{7}$$

The first quantity on the right is bounded in (6), and we now proceed to bound the second quantity.

Now for $1\leqslant i\leqslant i_1$, and $i_2\leqslant i\leqslant n$,

$$s(x)=s'_{i-1}H_3(\bar{x})+s_i'H_4(\bar{x}),\qquad x_{i-1}\leqslant x\leqslant x_i, \tag{8}$$

where $\bar{x}=x-x_{i-1}$ and where H_3 and H_4 are cubic polynomials given in [6, p. 212]. We will prove that there exists a constant K_1 such that $|s_i'|<K_1$ for $1\leqslant i\leqslant i_1$ and $i_2\leqslant i\leqslant n$. Then as in [7, p. 4] it will follow from (8) that

$$|s(x)|\leqslant K_1\,\|\,|H_3(x)|+|H_4(x)|\,\|_{[x_{i-1},x_i]}\leqslant\frac{K_1\delta_n}{4}. \tag{9}$$

If we show that $K_1=24(1+P)\max\{2/\overline{2-Q},3\}\|f\|_K$, then (9) coupled with (6)–(7) will establish (4).

Towards this end, one could solve for the unknown Hermite coordinates $\{s_i'\}_{i=1}^{n-1}$ of $s(x)$ via the well known relationship [1].

$$\begin{aligned}&h_{i+1}s'_{i-1}+2(h_i+h_{i+1})\,s_i'+h_is'_{i+1}\\&\quad=3\left\{\frac{h_{i+1}}{h_i}(s_i-s_{i-1})+\frac{h_i}{h_{i+1}}(s_{i+1}-s_i)\right\},\qquad i=1,2,\dots,n-1.\end{aligned}\tag{10}$$

As in [6], we write this system as

$$M\mathbf{X}=\mathbf{F} \tag{11}$$

where $\mathbf{X}=(s_1',s_2',\dots,s_{n-1}')^t$. Note that $[\mathbf{F}]_i=0$ (at least) for $i\leqslant i_1+2$ and $i\geqslant i_2-2$, if $\delta_n<a/16$.

Motivated by the method of proof in [7], we transform (11) so that the modified coefficient matrix has a bounded inverse and the slopes s_i', for $i\leqslant i_1$ and $i\geqslant i_2$, are bounded.

Let

$$M = \begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & M_{23} \\ 0 & M_{32} & M_{33} \end{bmatrix},$$

where M_{11} is $(i_1 + 2) \times (i_1 + 2)$, and M_{33} is $(n - i_2 + 2) \times (n - i_2 + 2)$. Further (in the notation of [4, p. 31]), define the elementary matrices

$$Q_i = E_{ii-1}(-m_{ii-1}/c_i), \qquad 1 \leqslant i \leqslant i_1 + 1,$$
$$R_i = E_{ii-1}(-m_{ii-1}/c_i), \qquad i_2 - 1 \leqslant i \leqslant n - 1.$$

Let $C = Q_1 \cdots Q_{i_1+1} R_{n-1} R_{n-2} \cdots R_{i_2-1}$; then

$$MC = \begin{bmatrix} M_{11}^{(1)} & M_{12} & 0 \\ M_{21} & M_{22} & M_{23} \\ 0 & M_{32} & M_{33}^{(1)} \end{bmatrix},$$

where

$$M_{11}^{(1)} = \begin{bmatrix} c_1 & & & 0 \\ h_3 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & h_{i_1+2} & c_{i_1+2} \end{bmatrix} \quad \text{and} \quad M_{33}^{(1)} = \begin{bmatrix} c_{i_2-2} & h_{i_2-1} & & \\ & \ddots & \ddots & \\ & & \ddots & h_{n-2} \\ 0 & & & c_{n-1} \end{bmatrix}$$

with

$$c_1 = m_{11}, \; c_{n-1} = m_{n-1,n-1},$$
$$c_i = m_{ii} - (h_{i+1} h_{i-1}/c_{i-1}), \qquad 2 \leqslant i \leqslant i_1 + 2,$$
$$c_{n-i} = m_{n-i,n-i} - (h_{n-i} h_{n-i+2}/c_{n-i+1}), \qquad 2 \leqslant i \leqslant n - i_2 + 2.$$

Next, define two diagonal matrices $D_1 = dg\{d_{ii}^{(1)}\}$ and $D_2 = dg\{d_{ii}^{(2)}\}$:

$$d_{ii}^{(1)} = 1, \quad 1 \leqslant i \leqslant i_1 + 2 \quad \text{and} \quad i_2 - 2 \leqslant i \leqslant n - 1$$
$$= 2(h_i + h_{i+1}), \quad i_1 + 3 \leqslant i \leqslant i_2 - 3,$$
$$d_{ii}^{(2)} = c_i, \quad 1 \leqslant i \leqslant i_1 + 2 \quad \text{and} \quad i_2 - 2 \leqslant i \leqslant n - 1$$
$$= 1, \quad i_1 + 3 \leqslant i \leqslant i_2 - 3.$$

Now (11) is equivalent to

$$(D_2^{-1} M C D_1^{-1})(D_1 C^{-1} \mathbf{X}) = D_2^{-1} \mathbf{F} = \mathbf{F}. \tag{12}$$

Let $N = D_2^{-1}MCD_1^{-1} = I + B$. Then, noting that $c_i > 2h_i + (3/2)h_{i+1}$ for $1 \leqslant i \leqslant i_1 + 2$, and $c_i > \frac{3}{2}h_i + 2h_{i+1}$ for $i_2 - 2 \leqslant i \leqslant n - 1$, we have

$$\|B\|_\infty \leqslant \max\{Q/2, \tfrac{2}{3}\} < 1.$$

Hence $\|N^{-1}\|_\infty < \max\{2/(2 - Q), 3\}$. This, coupled with

$$\|\mathbf{F}\|_\infty \leqslant 6(1 + P)\|f - g\|_K \leqslant 12(1 + P)\|f\|_K,$$

yields

$$\|D_1C^{-1}\mathbf{X}\|_\infty \leqslant 12 \max\left\{\frac{2}{2 - Q}, 3\right\}(1 + P)\|f\|_K \leqslant \frac{K_1}{2}. \qquad (13)$$

We now show that (13) implies $|s_i'| = |[X]_i| < K_1$, for $1 \leqslant i \leqslant i_1 + 1$ and $i_2 - 1 \leqslant i \leqslant n - 1$. From [4, p. 31],

$$Q_i^{-1} = E_{ii+1}(m_{ii+1}/m_{ii}),$$
$$R_i^{-1} = E_{ii-1}(m_{ii-1}/m_{ii});$$

so $C^{-1} = (p_{ij})$, where, by direct calculation, $p_{ij} = 0$ except

$$\begin{aligned} p_{ij} &= 1, \ i = j \\ &= m_{ii-1}/m_{ii}, \quad j = i + 1 \quad \text{and} \quad 1 \leqslant i \leqslant i_1 + 1 \\ &= m_{ii-1}/m_{ii}, \quad j = i - 1 \quad \text{and} \quad i_j - 1 \leqslant i \leqslant n - 1. \end{aligned}$$

We next note that $[D_1C^{-1}\mathbf{X}]_i = \mathbf{X}_i$ for $i = i_1 + 2, i_2 - 2$: hence from (13), $|s'_{i_1+2}| < K_1/2$ and $|s'_{i_2-2}| < K_1/2$. Now, since $[D_1C^{-1}\mathbf{X}]_i = [C^{-1}\mathbf{X}]_i$ for $1 \leqslant i \leqslant i_1 + 1$ and $i_2 - 1 \leqslant i \leqslant n - 1$, we have from the form of C^{-1} that

$$\left| s_i' + \frac{m_{ii+1}}{m_{ii}} s'_{i+1} \right| < K_1/2, \qquad \text{for} \quad 1 \leqslant i \leqslant i_1 + 1,$$

$$\left| s_i' + \frac{m_{ii-1}}{m_{ii}} s'_{i-1} \right| < K_1/2, \qquad \text{for} \quad i_2 - 1 \leqslant i \leqslant n - 1.$$

Therefore $|s_i'| < K_1$ for $1 \leqslant i \leqslant i_1 + 1$ and $i_2 - 1 \leqslant i \leqslant n - 1$, which establishes (9) and hence the theorem. Q.E.D.

COROLLARY 1. *Let f be a bounded function on $[-1, 1]$, with discontinuities at ξ_i, $1 \leqslant i \leqslant m$, but continuous elsewhere on $[-1, 1]$. Next, let $\{\Pi_n\}$ satisfy (1) and (2). If $a > 0$ is so chosen that*

$$\xi_j \notin [\xi_i - a/2, \xi_i + a/2]; \ i \neq j; \ i, j = 1, 2, \ldots, m, \qquad (14)$$

and if

$$K = \bigcup_{i=1}^{m} [\xi_i - a/4, \xi_i + a/4],$$

$$J = \overline{I - K},$$

$$L = \overline{\left\{[-1, 1] - \left\{\bigcup_{i=1}^{m} [\xi_i - a/2, \xi_i + a/2]\right\}\right\}},$$

and

$$\Delta_a = \max_{1 \leq i \leq m} |f(\xi_i + a/4) - f(\xi_i - a/4)|,$$

then

$$|(f - S_n f)(x)| \leq K_1\delta_n + K_2\{\omega_J(f, \delta_n) + 2\delta_n \Delta_a/a\} \tag{15}$$

for $x \in L$. *Hence, for* $x \neq \xi_i$, $1 \leq i \leq m$, $S_n f(x)$ *converges to* $f(x)$, *if* $\delta_n \to 0$.

Proof. The proof follows that of Theorem 1, with the new definitions for J, K, L, and Δ_a.

COROLLARY 2. *If the hypotheses of Corollary* 1 *are satisfied, and* f *is Lipschitz continuous on* $[-1, \xi_1), (\xi_1, \xi_2), (\xi_1, \xi_2),...,$ *and* $(\xi_m, 1]$, *with Lipschitz constants* $\mathscr{L}_i$, *respectively,* $i = 1, 2,..., m + 1$, *then*

$$|(f - S_n f)(x)| \leq [K_1 + (5/2)\mathscr{L}^*]\,\delta_n \tag{16}$$

for $|x| \geq a/2$, *where* $\mathscr{L}^* = \max\{\mathscr{L}_i + 2\Delta_a/a\}$. *Hence, for* $x \neq \xi_i$,

$$1 \leq i \leq m + 1,$$

$S_n f(x)$ *converges to* $f(x)$, *if* $\delta_n \to 0$.

Proof. We first note that f is Lipschitz continuous on $I_0 = I - \{\xi_j\}_{j=1}^m$, with Lipschitz constant $\mathscr{L}^*$. The proof follows that of Theorem 1, with the exception of the use of

$$\| g - S_n g \|_I \leq (5/2)\,\mathscr{L}^*\delta_n,$$

which was given in [7, Theorem 3].

Our final result concerns cubic spline approximation of continuous functions possessing continuous derivatives at all but a finite number of points.

THEOREM 2. *Let* $f \in C[-1, 1]$, $f' \in C(I_0)$, *where* $I_0 = [-1, 1] - \{\xi_j\}_{j=1}^m$. *Let* f' *be bounded on* I_0. *Let* $\{\Pi_n\}$ *be an* arbitrary *sequence of partitions of*

$[-1, 1]$, *with* $\delta_n \to 0$. *Then* $S_n f$ *converges to* f, *uniformly on* $[-1, 1]$. *In fact*,

$$\|S_n f - f\|_I \leq \tfrac{5}{2} \|f'\|_{I_0} \delta_n . \tag{17}$$

Proof. The Mean Value Theorem applies to each interval $[-1, \xi_1]$, $[\xi_1, \xi_2], \ldots, [\xi_m, 1]$, and hence, if x_1 and x_2 belong to the same interval, e.g. $[\xi_1, \xi_2]$, then $|f(x_1) - f(x_2)| \leq \|f'\|_{I_0} |x_1 - x_2|$.

Also if, say, $x_1 \in [\xi_1, \xi_2]$ and $x_2 \in [\xi_2, \xi_3]$, then

$$|f(x_1) - f(x_2)| \leq \|f'\|_{I_0} \{|x_1 - \xi_2| + |\xi_2 - x_2|\} = \|f'\|_{I_0} |x_1 - x_2|.$$

Hence f is Lipschitz on $[-1, 1]$ with Lipschitz constant $\|f'\|_{I_0}$.

But from [7, Theorem 3], we have (17).

In essence, Theorem 2 states that the order of convergence of a sequence of cubic spline interpolants to a function f is unchanged if the condition "$f \in C^1$" is replaced by "f is piecewise continuously differentiable."

In the spirit of the above discussion, a close look at the proofs of [3, Theorem 1] and [6, Theorem 1] indicates similar results when $f \in C^{m-1}[I]$, and when $f^{(m)}$ is piecewise continuous, $m = 2, 3$, or 4. To be more specific, relation (7) of [6] remains valid if $f \in C^1$ in each open interval (x_i, x_{i+1}): Taylor's formula with remainder still applies. This can be guaranteed simply by requiring all points of discontinuity of $f^{(1)}$ to be mesh points of each Π_n. A similar modification of the proof in [3] yields:

THEOREM 3. *Let* $m = 2, 3$, *or* 4, *let* $f \in C^{m-1}[-1, 1]$, *and let* $f^{(m)}$ *be continuous and bounded on* I_0. *Let* $\{\Pi_n\}$ *be a sequence of partitions of* $[-1, 1]$, *with* $\delta_n \to 0$, *such that each discontinuity of* $f^{(m)}$, ξ_j, *is a mesh point of* Π_n, $n = 1, 2, \ldots$. *Then* $(S_n f)^{(r)}$ *converges to* $f^{(r)}$, *uniformly on* $[-1, 1]$, $0 \leq r \leq m - 1$. *In fact*,

$$\|(S_n f - f)^{(r)}\|_I \leq \epsilon_{m,r} \|f^{(m)}\|_{I_0} h^{m-r}; \tag{18}$$

the constants $\epsilon_{m,r}$ *are given in* [3, *Table* 1].[2]

Remark. If $m = 4$ in (18), then $(S_n f - f)^{(3)}(x_i)$ is to be interpreted as a right- or left-hand derivative at the mesh point x_i.

3. EXAMPLES

We include two examples to illustrate the convergence results. The first is an example of spline approximation of a function with a discontinuity at the origin, and the second of a function with a discontinuity in slope at the origin.

[2] Since f' is defined at the endpoints, $S_n f$ is chosen so that $(S_n f)'(x_i) = f'(x_i)$, $i = 0, n$. See Examples 1 and 2.

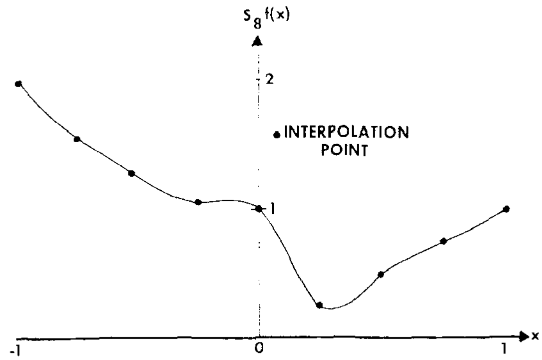

FIG. 1. Example 1 with $\delta_1 = 1/4$

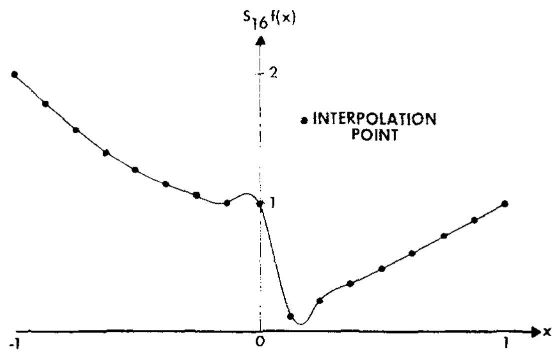

FIG. 2. Example 1 with $\delta_2 = 1/8$

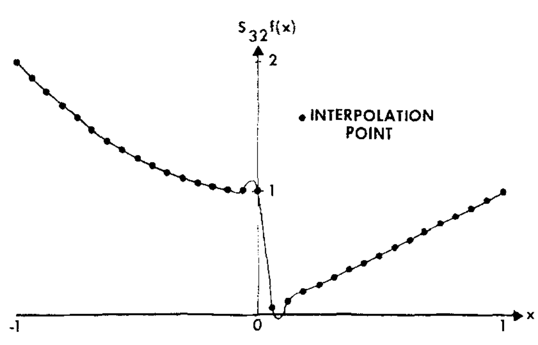

FIG. 3. Example 1 with $\delta_3 = 1/16$

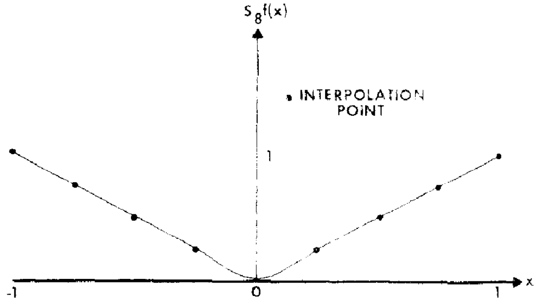

FIG. 4. Example 2 with $\delta_1 = 1/4$

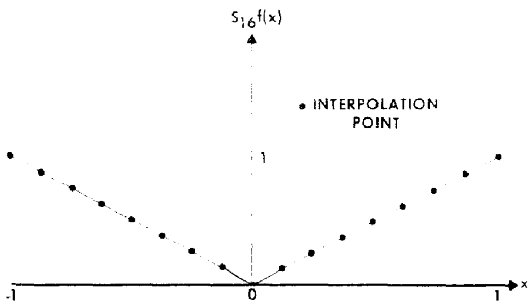

FIG. 5. Example 2 with $\delta_2 = 1/8$

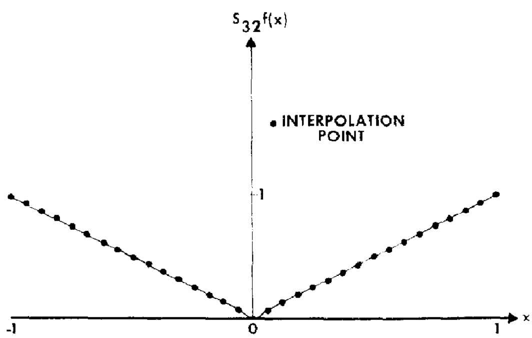

FIG. 6. Example 2 with $\delta_3 = 1/16$

EXAMPLE 1. $f(x) = x^2 + 1$, $-1 \leqslant x \leqslant 0$, and $f(x) = x$, $0 < x \leqslant 1$, with $\delta_1 = 1/4$, $\delta_2 = 1/8$, and $\delta_3 = 1/16$. See Figures 1, 2, and 3, respectively.

EXAMPLE 2. $f(x) = |x|$, $-1 \leqslant x \leqslant 1$, with $\delta_1 = 1/4$, $\delta_2 = 1/8$, and $\delta_3 = 1/16$. See Figures 4, 5, and 6, respectively.

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